

A note on f^\pm -Zagreb indices in respect of Jaco Graphs, $J_n(1)$, $n \in \mathbb{N}$ and the introduction of Khazamula irregularity

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Abstract

The topological graph indices $irr(G)$ related to the *first Zagreb index*, $M_1(G)$ and the *second Zagreb index*, $M_2(G)$ are of the oldest irregularity measures researched. Alberton [3] introduced the *irregularity* of G as $irr(G) = \sum_{e \in E(G)} imb(e)$, $imb(e) = |d(v) - d(u)|_{e=vu}$. In the paper of

Fath-Tabar [7], Alberton's indice was named the *third Zagreb indice* to conform with the terminology of chemical graph theory. Recently Ado et. al. [1] introduced the topological indice called *total irregularity*. The latter could be called the *fourth Zagreb indice*. We define the \pm Fibonacci weight, f_i^\pm of a vertex v_i to be $-f_{d(v_i)}$, if $d(v_i)$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. From the aforesaid we define the f^\pm -Zagreb indices. This paper presents introductory results for the undirected underlying graphs of Jaco Graphs, $J_n(1)$, $n \leq 12$. For more on Jaco Graphs $J_n(1)$ see [9, 10]. Finally we introduce the *Khazamula irregularity* as a new topological variant.

We also present five open problems.

Keywords: Total irregularity, Irregularity, Imbalance, Zagreb indices, \pm Fibonacci weight, Total f -irregularity, Fibonacci irregularity, f^\pm -Zagreb indices, Jaco graphs, Zeckendorf representation, Khazamula irregularity, Khazamula theorem.

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****On advice from arXiv Moderation this paper now incorporates similar ideas and variant results of another submission which has been removed.**

1 Introduction

The topological graph indices $irr(G)$ related to the *first Zagreb index*, $M_1(G) = \sum_{v \in V(G)} d^2(v) = \sum_{vu \in E(G)} (d(v) + d(u))$, and the *second Zagreb index*, $M_2(G) = \sum_{vu \in E(G)} d(v)d(u)$ are of the oldest irregularity measures researched. Alberton [3] introduced the *irregularity* of G as $irr(G) = \sum_{e \in E(G)} imb(e)$, $imb(e) = |d(v) - d(u)|_{e=vu}$. In the paper of Fath-Tabar [7], Alberton's indice was named the *third Zagreb indice* to conform with the terminology of chemical graph theory. Recently Ado et. al. [1] introduced the topological indice called *total irregularity* and defined it, $irr_t(G) = \frac{1}{2} \sum_{u, v \in V(G)} |d(u) - d(v)|$. The latter could be called the *fourth Zagreb indice*.

If the vertices of a simple undirected graph G on n vertices are labelled $v_i, i = 1, 2, 3, \dots, n$ then the respective definitions may be:

$$\begin{aligned} M_1(G) &= \sum_{i=1}^n d^2(v_i) = \sum_{i=1}^{n-1} \sum_{j=2}^n (d(v_i) + d(v_j))_{v_i u_j \in E(G)}, M_2(G) = \sum_{i=1}^{n-1} \sum_{j=2}^n d(v_i)d(v_j)_{v_i u_j \in E(G)}, \\ M_3(G) &= \sum_{i=1}^{n-1} \sum_{j=2}^n |d(v_i) - d(v_j)|_{v_i u_j \in E(G)} \text{ and } M_4(G) = irr_t(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |d(v_i) - d(v_j)| = \\ &= \sum_{i=1}^n \sum_{j=i+1}^n |d(v_i) - d(v_j)| \text{ or } \sum_{i=1}^{n-1} \sum_{j=i+1}^n |d(v_i) - d(v_j)|. \end{aligned}$$

For a simple graph on a singular vertex (1-empty graph), we define $M_1(G) = M_2(G) = M_3(G) = M_4(G) = 0$.

2 Zagreb indices in respect of \pm Fibonacci weights, f^\pm -Zagreb indices

We define the \pm Fibonacci weight, f_i^\pm of a vertex v_i to be $-f_{d(v_i)}$, if $d(v_i) = i$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. The f^\pm -Zagreb indices can now be defined as:

$$\begin{aligned} f^\pm Z_1(G) &= \sum_{i=1}^n (f_i^\pm)^2 = \sum_{i=1}^{n-1} \sum_{j=2}^n (|f_i^\pm| + |f_j^\pm|)_{v_i u_j \in E(G)}, f^\pm Z_2(G) = \sum_{i=1}^{n-1} \sum_{j=2}^n (f_i^\pm \cdot f_j^\pm)_{v_i u_j \in E(G)}, \\ f^\pm Z_3(G) &= \sum_{i=1}^{n-1} \sum_{j=2}^n |f_i^\pm - f_j^\pm|_{v_i u_j \in E(G)} \text{ and } f^\pm Z_4(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |f_i^\pm - f_j^\pm| = \sum_{i=1}^n \sum_{j=i+1}^n |f_i^\pm - f_j^\pm| \end{aligned}$$

or $\sum_{i=1}^{n-1} \sum_{j=i+1}^n |f_i^\pm - f_j^\pm|$. For a simple graph on a singular vertex (*1-empty graph*), we define $f^\pm Z_1(G) = f^\pm Z_2(G) = f^\pm Z_3(G) = f^\pm Z_4(G) = 0$.

2.1 Application to Jaco Graphs, $J_n(1), n \in \mathbb{N}$

For ease of reference some definitions in [9] are repeated. A particular family of finite directed graphs (*order 1*) called Jaco Graphs and denoted by $J_n(1), n \in \mathbb{N}$ are directed graphs derived from a particular well-defined infinite directed graph (*order 1*), called the *1-root digraph*. The *1-root digraph* has four fundamental properties which are; $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if v_k , for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

Definition 2.1. *The infinite Jaco Graph $J_\infty(1)$ is defined by $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$, [9].*

Definition 2.2. *The family of finite Jaco Graphs are defined by $\{J_n(1) \subseteq J_\infty(1) | n \in \mathbb{N}\}$. A member of the family is referred to as the Jaco Graph, $J_n(1)$, [9].*

Definition 2.3. *The set of vertices attaining degree $\Delta(J_n(1))$ is called the Jaconian vertices of the Jaco Graph $J_n(1)$, and denoted, $\mathbb{J}(J_n(1))$ or, $\mathbb{J}_n(1)$ for brevity, [9].*

From [9] we have *Bettina's Theorem*.

Theorem 2.1. *Let $\mathbb{F} = \{f_0, f_1, f_2, f_3, \dots\}$ be the set of Fibonacci numbers and let $n = f_{i_1} + f_{i_2} + \dots + f_{i_r}, n \in \mathbb{N}$ be the Zeckendorf representation of n . Then*

$$d^+(v_n) = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}.$$

Note: the degree of vertex v_i , denoted $d(v_i)$ refers to the degree in $J_\infty(1)$ hence $d(v_i) = i$. In the finite Jaco Graph the degree of vertex v_i is denoted $d(v_i)_{J_n(1)}$. The degree sequence is denoted $\mathbb{D}_n = (d(v_1)_{J_n(1)}, d(v_2)_{J_n(1)}, \dots, d(v_n)_{J_n(1)})$. By convention $\mathbb{D}_{i+1} = \mathbb{D}_i \cup d(v_{i+1})_{J_n(1)}$.

2.1.1 Algorithm to determine the degree sequence of a finite Jaco Graph, $J_n(1), n \in \mathbb{N}$.

Consider a finite Jaco Graph $J_n(1), n \in \mathbb{N}$ and label the vertices $v_1, v_2, v_3, \dots, v_n$.

Step 0: Set $n = n$. Let $i = j = 1$. If $j = n = 1$, let $\mathbb{D}_i = (0)$ and go to Step 6, else set $\mathbb{D}_i = \emptyset$ and go to Step 1.

Step 1: Determine the j^{th} Zeckendorf representation say, $j = f_{i_1} + f_{i_2} + \dots + f_{i_r}$, and go to Step 2.

Step 2: Calculate $d^+(v_j) = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}$, then go to Step 3.

Step 3: Calculate $d^-(v_j) = j - d^+(v_j)$, and let $d(v_j) = d^+(v_j) + d^-(v_j)$, then go to Step 4.

Step 4: If $d(v_j) \leq n$, set $d(v_j)_{J_n(1)} = d(v_j)$ else, set $d(v_j)_{J_n(1)} = d^-(v_j) + (n - j)$ and set $\mathbb{D}_j = \mathbb{D}_i \cup d(v_j)_{J_n(1)}$ and go to Step 5.

Step 5: If $j = n$ go to Step 6 else, set $i = i + 1$ and $j = i$ and go to Step 1.

Step 6: Exit.

2.1.2 Tabled values of $\mathbb{F}^\pm(J_n(1))$, for finite Jaco Graphs, $J_n(1), n \leq 12$.

For illustration the adapted table below follows from the Fisher Algorithm [9] for $J_n(1), n \leq 12$. Note that the Fisher Algorithm determines $d^+(v_i)$ on the assumption that the Jaco Graph is always sufficiently large, so at least $J_n(1), n \geq i + d^+(v_i)$. For a smaller graph the degree of vertex v_i is given by $d(v_i)_{J_n(1)} = d^-(v_i) + (n - i)$. In [9] Bettina's theorem describes an arguably, closed formula to determine $d^+(v_i)$. Since $d^-(v_i) = n - d^+(v_i)$ it is then easy to determine $d(v_i)_{J_n(1)}$ in a smaller graph $J_n(1), n < i + d^+(v_i)$. The f_i^\pm -sequence of $J_n(1)$ is denoted $\mathbb{F}^\pm(J_n(1))$.

Table 1.

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = i - d^-(v_n)$	$\mathbb{F}^\pm(J_i(1))$
1	0	1	(0)
2	1	1	(-1, -1)
3	1	2	(-1, 1, -1)
4	1	3	(-1, 1, 1, -1)
5	2	3	(-1, 1, -2, 1, 1)
6	2	4	(-1, 1, -2, -2, -2, 1)
7	3	4	(-1, 1, -2, 3, 3, -2, -2)
8	3	5	(-1, 1, -2, 3, -5, 3, 3, -2)
9	3	6	(-1, 1, -2, 3, -5, -5, -5, 3, -2)
10	4	6	(-1, 1, -2, 3, -5, 8, 8, -5, 3, 3)
11	4	7	(-1, 1, -2, 3, -5, 8, -13, 8, -5, -5, 3)
12	4	8	(-1, 1, -2, 3, -5, 8, -13, -13, 8, 8, -5, 3)

Since it is known that a sequence $(d_1, d_2, d_3, \dots, d_n)$ of non-negative integers is a degree sequence of some graph G if and only if $\sum_{i=1}^n d_i$ is even. It implies that a degree sequence

has an even number of *odd entries*. Hence, we know that the f_i^\pm -sequence of $J_n(1)$ denoted, $\mathbb{F}^\pm(J_n(1)), n \in \mathbb{N}$ has an *even number* of, $-f_{d(v_i)}$ entries. Following from Table 1 the table below depicts the values $f^\pm Z_1(J_n(1)), f^\pm Z_2(J_n(1)), f^\pm Z_3(J_n(1))$ and $f^\pm Z_4(J_n(1))$ for $J_n(1), n \leq 12$.

Table 2.

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	$f^\pm Z_1(J_i(1))$	$f^\pm Z_2(J_i(1))$	$f^\pm Z_3(J_i(1))$	$f^\pm Z_4(J_i(1))$
1	0	1	0	0	0	0
2	1	1	2	1	0	0
3	1	2	3	-2	4	4
4	1	3	4	-1	4	8
5	2	3	8	-6	11	16
6	2	4	15	5	11	25
7	3	4	32	-26	35	56
8	3	5	62	-19	50	98
9	3	6	103	0	72	138
10	4	6	211	38	119	251
11	4	7	396	-238	210	402
12	4	8	604	-158	273	566

3 Khazamula irregularity

Let G^\rightarrow be a simple directed graph on $n \geq 2$ vertices labelled $v_1, v_2, v_3, \dots, v_n$. Let all vertices v_i carry its \pm Fibonacci weight, f_i^\pm related to $d(v_i) = d(v^+(v_i)) + d^-(v_i)$. Also let vertex v_j be a head vertex of v_i and choose any $d(v_i^h) = \max(d(v_j)_{\forall v_j})$.

Definition 3.1. Let G^\rightarrow be a simple directed graph on $n \geq 2$ vertices with each vertex carrying its \pm Fibonacci weight, f_i^\pm . For the function $f(x) = mx + c, x \in \mathbb{R}$ and $m, c \in \mathbb{Z}$ we define the Khazamula irregularity as:

$$irr_k(G^\rightarrow) = \sum_{i=1}^n \left| \int_{f_i^\pm}^{d(v_i^h)} f(x) dx \right|.$$

Note: Vertices v with $d^+(v) = 0$, are *headless* and the corresponding integral terms to the summation are defined *zero*. Hence, $irr_k(K_1^\rightarrow) = 0$.

Let G be a simple connected undirected graph on n vertices which are labelled, $v_1, v_2, v_3, \dots, v_n$. Also let G have ϵ edges. It is known that G can be *orientated* in 2^ϵ ways, including the cases of isomorphism. Finding the relationship between the different values of $irr_k(G^\rightarrow)$ and $irr_k^c(G^\rightarrow)$ (to follow in subsection 3.3) in respect of the different orientations for G in general is stated as an open problem. In this section we give results in respect of particular orientations of paths, cycles, wheels and complete bipartite graphs.

3.1 irr_k for Paths, Cycles, Wheels and Complete Bipartite Graphs

Proposition 3.1. For a directed path $P_n^\rightarrow, n \geq 2$ which is consecutively directed from left to right we have that the Khazamula irregularity, $irr_k(P_n^\rightarrow) = \lfloor \frac{3}{2}(n-2)m + nc \rfloor$.

Proof. Label the vertices of the directed path P_n^\rightarrow consecutively from left to right $v_1, v_2, v_3, \dots, v_n$.

From the definition $irr_k(P_n^\rightarrow) = \sum_{i=1}^n \left| \int_{f_i^\pm}^{d(v_i^h)} f(x) dx \right|$, it follows that we have:

$$\sum_{i=1}^n \left| \int_{f_i^\pm}^{d(v_i^h)} f(x) dx \right| = \left| \int_{-1}^2 f(x) dx + \underbrace{\int_1^2 f(x) dx + \dots + \int_1^2 f(x) dx}_{(n-3)\text{-terms}} + \int_1^1 f(x) dx \right|.$$

So we have, $\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |(\frac{1}{2}mx^2 + cx)|_{-1}^2 + (n-3)(\frac{1}{2}mx^2 + cx)|_1^2 + 0| =$
 $|2m + 2c - \frac{1}{2}m + c + (n-3)(2m + 2c - \frac{1}{2}m - c)| = |\frac{3}{2}m + 3c + \frac{3}{2}(n-3)m + (n-3)c| =$
 $|\frac{3}{2}(n-2)m + nc|.$ \square

Proposition 3.2. *For a directed cycle C_n^\rightarrow which is consecutively directed clockwise we have that the Khazamula irregularity, $irr_k(C_n^\rightarrow) = n|\frac{3}{2}m + c|.$*

Proof. Label the vertices of the directed cycle C_n^\rightarrow consecutively clockwise $v_1, v_2, v_3, \dots, v_n$. So vertices carry the \pm Fibonacci weight, $f_{i \forall i}^\pm = f_1 = 1$. Also a head vertex is always unique with degree = 2. From the definition $irr_k(C_n^\rightarrow) = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$, it follows that we have:

$$\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = \underbrace{|\int_1^2 f(x)dx + \int_1^2 f(x)dx + \dots + \int_1^2 f(x)dx|}_{n\text{-terms}} = |n(\frac{1}{2}mx^2 + cx)|_1^2 =$$

$$|n(2m + 2c - \frac{1}{2}m - c)| = |n(\frac{3}{2}m + c)| = n|\frac{3}{2}m + c|. \quad \square$$

Proposition 3.3. *For a directed Wheel graph $W_{(1,n)}^\rightarrow$ with the axle vertex u_1 and the wheel vertices v_1, v_2, \dots, v_n and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise v_1, v_2, \dots, v_n , we have that:*

$$irr_k(W_{(1,n)}^\rightarrow) \begin{cases} = |\frac{(5n-f_n^2+9)}{2}m + (5n - f_n + 3)c|, & \text{if } n \text{ is even,} \\ = |\frac{(5n-f_n^2+9)}{2}m + (5n + f_n + 3)c|, & \text{if } n \text{ is uneven.} \end{cases}$$

Proof. Consider a Wheel graph $W_{(1,n)}^\rightarrow$ with the axle vertex u_1 and the wheel vertices v_1, v_2, \dots, v_n and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise v_1, v_2, \dots, v_n .

Case 1: If n is even then $d(u_1)$ is even and carries the \pm Fibonacci weight, f_n . Obviously the wheel vertices have $d(v_i) = 3_{\forall i}$, hence carry the \pm Fibonacci weight, $f_3 = -2_{\forall v_i}$. So from the definition of the Khazamula irregularity we have that:

$$irr_k(W_{(1,n)}^\rightarrow) = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |n \int_{-2}^3 f(x)dx + \int_{f_n}^3 f(x)dx| \text{ if } n \text{ is even. This results}$$

$$\text{in, } irr_k = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |n(\frac{9}{2}m + 3c - 2m + 2c) + (\frac{9}{2}m + 3c - \frac{f_n^2}{2}m - f_n c)| =$$

$$|\frac{5}{2}nm + 5nc + \frac{9}{2}m + 3c - \frac{f_n^2}{2}m - f_nc| = |\frac{(5n-f_n^2+9)}{2}m + (5n - f_n + 3)c|.$$

Case 2: If n is uneven then $d(u_1)$ is uneven and carries the \pm Fibonacci weight, $-f_n$. So in the Riemann integral $\int_{-f_n}^3 f(x)dx$ we have $(\frac{9}{2}m + 3c - \frac{f_n^2}{2}m + f_nc)$. So the result $irr_k(W_{(1,n)}^\rightarrow) = |\frac{(5n-f_n^2+9)}{2}m + (5n + f_n + 3)c|$ if n is uneven, follows. \square

Consider the complete bipartite graph $K_{(n,m)}$ and call the n vertices the *left-side vertices* and the m vertices the *right-side vertices*. Orientate $K_{(n,m)}$ strictly from *left-side vertices* to *right-side vertices* to obtain $K_{(n,m)}^{l \rightarrow r}$.

Proposition 3.4. *For the directed graph $K_{(n,m)}^{l \rightarrow r}$ we have that:*

$$irr_k(K_{(n,m)}^{l \rightarrow r}) \begin{cases} = |\frac{(n^3-nf_m^2)}{2}m + (n^2 - nf_m)c|, & \text{if } m \text{ is even,} \\ = |\frac{(n^3-nf_m^2)}{2}m + (n^2 + nf_m)c|, & \text{if } m \text{ is uneven.} \end{cases}$$

Proof. For the directed graph $K_{(n,m)}^{l \rightarrow r}$ we have that all *left-side vertices* say v_1, v_2, \dots, v_n have $d^+(v_i) = m$, whilst all *right-side vertices* say u_1, u_2, \dots, u_m have $d^-(u_i) = n$ and $d^+(u_i) = 0$.

Case 1: If m is even it follows from the definition that, $irr_k(K_{(n,m)}^{l \rightarrow r}) = n|\int_{f_m}^n f(x)dx|$. So we have that $irr_k(K_{(n,m)}^{l \rightarrow r}) = n|(\frac{1}{2}mx^2 + cx)|_{f_m}^n| = n|\frac{n^2}{2}m + nc - (\frac{f_m^2}{2}m + f_mc)| = |\frac{(n^3-nf_m^2)}{2}m + (n^2 - nf_m)c|$.

Case 2: If m is uneven the *left-side vertices* all carry the \pm Fibonacci weight, $-f_m$. Hence, the result follows as in Case 1, accounting for $-f_m$. \square

Example problem 1: Let $n = 1$ or 5 and $f(x) = mx$. Prove that $irr_k(K_{(1,n)}^\rightarrow) = 0$ or $|12m|$ and,

$$irr_k(K_{(1,n)}^{l \rightarrow r}) \begin{cases} = 0 \\ or \\ = 5(irr_k(K_{(1,n)}^\rightarrow)) = 60|m|. \end{cases}$$

Proof. Let $n = 1$ and let $f(x) = mx$. From the definition of $irr_k(G^\rightarrow)$ it follows that $irr_k(K_{(1,n)}^\rightarrow) = \int_{-1}^1 |mx \cdot dx|_{for-v_1} = |\frac{1}{2}mx^2|_{-1}^1| = 0$. We also have that $irr_k(K_{(1,n)}^\rightarrow) = \int_{-1}^1 |mx \cdot dx|_{for-u_1} = |\frac{1}{2}mx^2|_{-1}^1| = 0$.

Let $n = 5$ and let $f(x) = mx$. Now we have that $irr_k(K_{(1,n)}^\rightarrow) = \int_{-5}^1 |mx \cdot dx|_{for-v_1} = |\frac{1}{2}mx^2|_{-5}^1| = |12m|$.

For $irr_k(K_{(1,n)}^\rightarrow)$ we have $\sum_{i=1}^5 \int_{-1}^5 |mx \cdot dx|_{for-u_i, i=1,2,\dots,5} = 5(\int_{-1}^5 |mx \cdot dx|) = 5|\frac{1}{2}mx^2|_{-1}^5| = 5|12m| = 60|m|$. \square

3.2 Khazamula's Theorem

Consider two simple connected directed graphs, G^\rightarrow and H^\rightarrow . Let the vertices of G^\rightarrow be labelled v_1, v_2, \dots, v_n and the vertices of H^\rightarrow be labelled u_1, u_2, \dots, u_m . Define the *directed join* as $(G^\rightarrow + H^\rightarrow)^\rightarrow$ conventionally, with the arcs $\{(v_i, u_j) | \forall v_i \in V(G^\rightarrow), u_j \in V(H^\rightarrow)\}$.

Theorem 3.5. *Consider two simple connected directed graphs, G^\rightarrow on n vertices and H^\rightarrow on m vertices then, $irr_k((G^\rightarrow + H^\rightarrow)^\rightarrow) = |n \int_{f_i^\pm|_{v_i \in V((G^\rightarrow + H^\rightarrow)^\rightarrow)}^{\Delta(H^\rightarrow)+n}} f(x)dx + \sum_{i=1}^m \int_{f_{d(u_i)+1}}^{d(u_i^h)+1} f(x)dx|$.*

Proof. Note that in the graph G^\rightarrow the maximum degree $\Delta(G^\rightarrow) = \max(d^+(v_\ell) + d^-(v_\ell)) \leq n - 1$ for at least one vertex v_ℓ . If such a vertex v_ℓ is indeed the *head vertex* of a vertex v_t , then $\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$, will contain the term $\int_{f_t^\pm}^{\Delta(G)} f(x)dx$.

In H^\rightarrow the maximum degree $\Delta(H^\rightarrow) = \max(d^+(u_s) + d^-(u_s)) \geq 1$ for some vertex u_s . Hence, in the directed graph $(G^\rightarrow + H^\rightarrow)^\rightarrow$, all terms of $\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$ reduces to *zero* and are replaced by the terms $\int_{f_i^\pm|_{v_i \in V((G^\rightarrow + H^\rightarrow)^\rightarrow)}^{\Delta(H^\rightarrow)+n}} f(x)dx$, because $\Delta(G^\rightarrow) \leq n - 1 < \Delta(H^\rightarrow) + n$.

In respect of H^\rightarrow we have that each $d(u_i)_{\forall i}$ increases by exactly 1 so the value of $f_{d(u_i)_{\forall i}}$ switches between \pm and adopts the value $f_{d(u_i)+1}$. Similarly all *head vertices'* degree increases by exactly 1. These observations result in:

$$irr_k((G^\rightarrow + H^\rightarrow)^\rightarrow) = |n \int_{f_i^\pm|_{v_i \in V((G^\rightarrow + H^\rightarrow)^\rightarrow)}^{\Delta(H^\rightarrow)+n}} f(x)dx + \sum_{i=1}^m \int_{f_{d(u_i)+1}}^{d(u_i^h)+1} f(x)dx|. \quad \square$$

Example problem 2: An application of the Khazamula theorem to the graph $(C_n^\rightarrow + K_1)^\rightarrow$ in respect of $f(x) = mx$, results in $irr_k((C_n^\rightarrow + K_1)^\rightarrow) = \frac{1}{3}(n^2 - 4)irr_k(C_n^\rightarrow)|_{f(x)=mx}$.

3.3 Khazamula c-irregularity for orientated Paths, Cycles, Wheels and Complete Bipartite Graphs

Let $f(x) = \sqrt{r^2 - x^2}, x \in \mathbb{R}$ and $r = \max\{d(v_i)_{\forall v_i, d^-(v_i) \geq 1}, \text{ or } |(f_i^\pm)|_{\forall v_i}\}$. We define *Khazamula c-irregularity* as $irr_k^c(G^\rightarrow) = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$. It is known that $\int_a^b \sqrt{r^2 - x^2}dx = (\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\frac{x}{r})|_a^b$. Also note that $\arcsin\theta$ applies to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ to ensure a singular value for the respective integral terms.

Proposition 3.6. *For a directed path $P_n^\rightarrow, n \geq 3$ which is consecutively directed from left to right we have that the Khazamula c-irregularity, $irr_k^c(P_n^\rightarrow) = (n - 2)(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$.*

Proof. Label the vertices of the directed path $P_n^\rightarrow, n \geq 3$ consecutively from left to right $v_1, v_2, v_3, \dots, v_n$. Note that $r = \max\{d(v_i)_{\forall v_i}, \text{ or } |(f_i^\pm)|_{\forall v_i}\} = 2$. From the definition $irr_k^c(P_n^\rightarrow) = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$, it follows that we have:

$$\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |\int_{-1}^2 f(x)dx + \underbrace{\int_1^2 f(x)dx + \dots + \int_1^2 f(x)dx}_{(n-3)\text{-terms}} + \int_1^1 f(x)dx|.$$

So we have, $\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |(\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\frac{x}{r})|_{-1}^2 + (n - 3)|(\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\frac{x}{r})|_1^2 + |(\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\frac{x}{r})|_1^1| = |(\frac{1}{2}x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2})|_{-1}^2 + (n - 3)|(\frac{1}{2}x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2})|_1^2| = |(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}) + (n - 3)(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})| = |(n - 2)(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})| = (n - 2)(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}). \quad \square$

Proposition 3.7. *For a directed cycle C_n^\rightarrow which is consecutively directed clockwise we have that the Khazamula c-irregularity, $irr_k^c(C_n^\rightarrow) = n(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$.*

Proof. Label the vertices of the directed cycle C_n^\rightarrow consecutively clockwise $v_1, v_2, v_3, \dots, v_n$. So all vertices carry the \pm Fibonacci weight, $f_{i_{v_i}}^\pm = f_1 = 1$. Also a head vertex is always

unique with degree = 2. So $r = \max\{d(v_i)_{\forall v_i}, \text{or } |(f_i^\pm)|_{\forall v_i}\} = 2$. From the definition $\text{irr}_k(C_n^\rightarrow) = \sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$, it follows that we have:

$$\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx| = |\underbrace{\int_1^2 f(x)dx + \int_1^2 f(x)dx + \cdots + \int_1^2 f(x)dx}_{n\text{-terms}}| = n|(\frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\frac{x}{2})|_1^2| =$$

$$n|(0 + 2\arcsin 1 - \frac{\sqrt{3}}{2} - 2\arcsin\frac{1}{2})| = n|(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})| = n(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}). \quad \square$$

Proposition 3.8. For a directed Wheel graph $W_{(1,n)}^\rightarrow$ with the axle vertex u_1 and the wheel vertices v_1, v_2, \dots, v_n and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise v_1, v_2, \dots, v_n , we have that:

$$\text{irr}_k(W_{(1,n)}^\rightarrow) \begin{cases} = 4\sqrt{5} + 9\pi + 18\arcsin\frac{2}{3}, & \text{if } n = 3 \text{ or } 4, \\ = |\frac{3}{2}(n+1)\sqrt{f_n^2-9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} - \frac{f_n^2\pi}{4} + A|, & \text{if } n \geq 6 \text{ and even,} \\ = |\frac{3}{2}(n+1)\sqrt{f_n^2-9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \frac{f_n^2\pi}{4} + B|, & \text{if } n \geq 5 \text{ and uneven,} \end{cases}$$

with: $A = n(\sqrt{f_n^2-4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n})$ and $B = n(\sqrt{f_n^2-4} - \frac{f_n^2}{2}\arcsin\frac{2}{f_n})$.

Proof. Consider a Wheel graph $W_{(1,n)}^\rightarrow$ with the axle vertex u_1 and the wheel vertices v_1, v_2, \dots, v_n and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise v_1, v_2, \dots, v_n .

$$\text{Case 1: If } n = 3 \text{ we have that } \text{irr}_k^c(W_{(1,3)}^\rightarrow) = |\underbrace{\int_{-2}^3 \sqrt{9-x^2}dx}_{\text{for } u_1} + 3 \underbrace{\int_{-2}^3 \sqrt{9-x^2}dx}_{\text{for } v_i}|, i = 1, 2, 3.$$

$$\text{Therefore, } \text{irr}_k^c(W_{(1,3)}^\rightarrow) = 4|(\int_{-2}^3 \sqrt{9-x^2}dx)| = 4|(\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\arcsin\frac{x}{3})|_{-2}^3| = 4|(\frac{9}{2}\arcsin 1 -$$

$$(-\sqrt{9-4} - \frac{9}{2}\arcsin\frac{2}{3}))| = 4\sqrt{5} + 9\pi + 18\arcsin\frac{2}{3}.$$

$$\text{If } n = 4 \text{ then } \text{irr}_k^c(W_{(1,4)}^\rightarrow) = |\underbrace{\int_3^3 \sqrt{9-x^2}dx}_{\text{for } u_1} + 4 \underbrace{\int_{-2}^3 \sqrt{9-x^2}dx}_{\text{for } v_i}|, i = 1, 2, 3, 4. \text{ Hence, the}$$

result follows.

Case 2: If $n \geq 6$ and *even* we have $irr_k^c(W_{(1,n)}^\rightarrow) = \underbrace{|\int_{f_n}^3 \sqrt{f_n^2 - x^2} dx|}_{for, u_1} + n \underbrace{|\int_{-2}^3 \sqrt{f_n^2 - x^2} dx|}_{for, v_i}, i = 1, 2, \dots, n$. So we have $irr_k^c(W_{(1,n)}^\rightarrow) = |(\frac{1}{2}x\sqrt{f_n^2 - x^2} + \frac{f_n^2}{2}\arcsin\frac{x}{f_n})|_{f_n}^3 + n(\frac{1}{2}x\sqrt{f_n^2 - x^2} + \frac{f_n^2}{2}\arcsin\frac{x}{f_n})|_{-2}^3| = |\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - (\frac{f_n}{2}\sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2}\arcsin 1) + n(\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - (-\sqrt{f_n^2 - 4} - \frac{f_n^2}{2}\arcsin\frac{2}{f_n}))| = |\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - (\frac{f_n}{2}\sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2}\arcsin 1) + n(\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \sqrt{f_n^2 - 4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n})| = |\frac{3}{2}(n+1)\sqrt{f_n^2 - 9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \frac{f_n^2\pi}{4} + A|$, with $A = n(\sqrt{f_n^2 - 4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n})$.

Case 3: Similar to Case 2 and accounting for $n \geq 5$ and *uneven*. □

Consider the complete bipartite graph $K_{(n,m)}$ and call the n vertices the *left-side vertices* and the m vertices the *right-side vertices*. Orientate $K_{(n,m)}$ strictly from *left-side vertices* to *right-side vertices* to obtain $K_{(n,m)}^{l \rightarrow r}$.

Proposition 3.9. *For the directed graph $K_{(n,m)}^{l \rightarrow r}$ we have that:*

$$irr_k^c(K_{(n,m)}^{l \rightarrow r}) \begin{cases} = |\frac{n^2\pi}{4} - A|, & \text{if } n \geq f_m \text{ and } m \text{ is even,} \\ = |\frac{n^2\pi}{4} + A|, & \text{if } n \geq f_m \text{ and } m \text{ is uneven,} \\ = |B - \frac{f_m^2\pi}{4}|, & \text{if } f_m > n \text{ and } m \text{ is even,} \\ = |B + \frac{f_m^2\pi}{4}|, & \text{if } f_m > n \text{ and } m \text{ is uneven,} \end{cases}$$

with $A = \frac{f_m}{2}\sqrt{n^2 - f_m^2} + \frac{n^2}{2}\arcsin\frac{f_m}{n}$ and $B = \frac{n}{2}\sqrt{f_m^2 - n^2} + \frac{f_m^2}{2}\arcsin\frac{n}{f_m}$.

Proof. For the directed graph $K_{(n,m)}^{l \rightarrow r}$ we have that all *left-side vertices* say v_1, v_2, \dots, v_n have $d^+(v_i) = m$, whilst all *right-side vertices* say u_1, u_2, \dots, u_m have $d^-(u_i) = n$ and $d^+(u_i) = 0$.

Case 1: Since $d^+(u_i) = 0, \forall i$ the terms in $\sum_{i=1}^n |\int_{f_i^\pm}^{d(v_i^h)} f(x)dx|$, stem from vertices $v_i, \forall i$ only. Furthermore, since $r = \max\{d(u_i)_{\forall i, d^-(u_i) \geq 1}, \text{or } f_m\}$ and $n \geq f_m$, we have $r = n$.

It follows that $irr_k^c(K_{(n,m)}^{l \rightarrow r}) = n |\int_{f_m}^n \sqrt{n^2 - x^2} dx| = |(\frac{1}{2}x\sqrt{n^2 - x^2} + \frac{n^2}{2}\arcsin \frac{x}{n})|_{f_m}^n| = |\frac{n^2}{2}\arcsin 1 - (\frac{f_m}{2}\sqrt{n^2 - f_m^2} + \frac{n^2}{2}\arcsin \frac{f_m}{n})| = |\frac{n^2\pi}{4} - A|$, with $A = \frac{f_m}{2}\sqrt{n^2 - f_m^2} + \frac{n^2}{2}\arcsin \frac{f_m}{n}$.

Case 2: Similar to Case 1 and accounting for m is *uneven*.

Case 3: Similar to Case 1 and accounting for $f_m > n$, m is *even*.

Case 4: Similar to Case 1 and accounting for $f_m > n$, m is *uneven*. □

[Open problem: If possible, generalise Khazamula's irregularity for simple directed graphs.]

[Open problem: Find a closed or, recursive formula for $f^\pm Z_1(J_n(1)), f^\pm Z_2(J_n(1)), f^\pm Z_3(J_n(1))$, and $f^\pm Z_4(J_n(1))$.]

[Open problem: Where possible, describe the terms of the Khazamula theorem in terms of $irr_k(G^\rightarrow)$ and $irr_k(H^\rightarrow)$ for specialised classes of simple directed graphs.]

[Open problem: If possible, formulate and prove *Khazamula's c-Theorem* related to *Khazamula c-irregularity* for simple directed graphs in general.]

[Open problem: Let G be a simple connected undirected graph on n vertices labelled, $v_1, v_2, v_3, \dots, v_n$. Also let G have ϵ edges. It is known that G can be *orientated* in 2^ϵ ways, including the cases of isomorphism. Find the relationship between the different values of $irr_k(G^\rightarrow)$ in respect of the different orientations.]

[Open problem: Let G be a simple connected undirected graph on n vertices labelled, $v_1, v_2, v_3, \dots, v_n$. Also let G have ϵ edges. It is known that G can be *orientated* in 2^ϵ ways, including the cases of isomorphism. Find the relationship between the different values of $irr_k^c(G^\rightarrow)$ in respect of the different orientations.]

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